SOLVING A CONJECTURE ABOUT CERTAIN $f$ - EXPANSIONS

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The conjecture asserts that the equivalence of the label sequence of the regular continued fraction (RCF) expansion to the sequence $(\xi_n)_{n \in \mathbb{N}}$, associated with it by the basic existence Theorem 1.1.2 from [6], still holds for the label sequence of any $f$-expansion satisfying conditions (C) and (BD$^{(2)}$). We prove that condition (C) and a strengthening of a Lipschitz condition used in [8] are sufficient to ensure a necessary and sufficient condition under which the asserted equivalence holds. The proof involves processes on several probability spaces and some associated dynamical systems relating the $f$-expansion considered to $\text{r.v.s.}$ on the probability space used in the concluding theorem.

1. INTRODUCTION

Let $\mathbb{N}_* = \{1, 2, \ldots \}$ and $\mathbb{N} = \mathbb{N}_* \cup \{0\}$. Given an RSCC $(W, X, u, P, W, P)$, where $X$ is a countable set, by Theorem 1.1.2 in [6] for any $w \in W$ there exist a probability space $(\Omega, K, \mathcal{P}_w)$ and a sequence $(\xi_n)_{n \in \mathbb{N}_*}$ of $X$-valued r.v.s. on $(\Omega, K)$ such that

\begin{enumerate}
  \item $P_w(\xi_1 = i) = P(w, i)$, $P_n(\xi_{n+1} = i | \xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_n) = P(\zeta_n, i)$, \quad $i \in X$, $n \in \mathbb{N}_*$,
\end{enumerate}

where

$\zeta_n = u_{\xi_1, \ldots, \xi_n}(w)$, \quad $n \in \mathbb{N}_*$, \quad $\zeta_0 \equiv w$.

b. The sequence $(\xi_n)_{n \in \mathbb{N}}$ is a $W$-valued homogenous Markov chain.

In particular, this theorem holds for the RSCC associated with the RCF and $D$-adic expansions. Denoting by $(a_n)_{n \in \mathbb{N}}$ the label sequence and by $\lambda$ the Lebesgue measure, in these two cases the following equations which can be obtained by direct computation do hold:

$\lambda(a_1 = i) = P(0, i)$; \quad $\lambda(a_{n+1} = i | a_1 = i_1, \ldots, a_n = i_n) = P(u_{a_n, \cdot}(0))$

In what follows we shall use the notation from [6] where is shown that with any $f$-expansion satisfying conditions (BD$^{(2)}$) and (C) one can associate an RSCC. Hence one may particularize (1) to such $f$-expansions.

Let $I = [0, 1]$ and denote by $B_I$ the $\sigma$-algebra of Borel sets in $I$. For any $n \in \mathbb{N}_*$ let $I(i^{(n)})$, $i^{(n)} \in X^n$, be the fundamental intervals of order $n$ and $E_n$ the set of their endpoints. Let be $[\alpha, \beta]$ the interval of definition of $f$ and put $I := I \setminus \cup_{n \geq 1} E_n$. Clearly, $\lambda(I) = 1$. 

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It is known (see, e.g., [6]) that when \( f \) has a second derivative in \( [\alpha, \beta] \setminus \mathbb{N} \), and conditions (C) and \((\text{BD})^2\) are fulfilled, hence an RSCC can be associated, the following properties hold.

c. The terms of the label sequence \( (a_n)_{n \in \mathbb{N}} \) of the \( f \)-expansion are r.v. on \((I, B_I)\). Hence an \( f \)-expansion exists for almost every point \( w \in I \).

d. The so-called representation map \( \varphi(w) = (a_n(w))_{n \in \mathbb{N}} \) is well defined in \( I \). Hence an \( f \)-expansion exists for almost every point \( w \in I \).

e. The term sequence \( (a_n)_{n \in \mathbb{N}} \) of the \( f \)-expansion is r.v. on \((I, B_I)\). Hence an \( f \)-expansion exists for almost every point \( w \in I \).

2. AN INFINITE ORDER CHAIN REPRESENTATION

We define the natural extension \( T \) of \( \tau_f \) by \( T(\theta, \omega) = (\tau_f(\theta), s^n(\omega)) \). This is a one-to-one transformation of \( I \times I \) with inverse \( T^{-1}(\theta, \omega) = (s^n(\omega), \tau_f(\omega)) \). Let us denote \( t^{(n)} := (i_n, \ldots, i_1) \in X^n, n \in \mathbb{N} \).

We now define constructively a \( T \)-invariant measure. Let \( n, s \in \mathbb{N} \), \( i^{(n)} \in X^n, v^{(s)} \in X^s \). Since \( (\tilde{t}^{(n)} \tilde{v}^{(s)}) = (i_n, \ldots, i_1, v_1, \ldots, v_s) \), we have

\[
    T' \left( I \left( \tilde{v}^{(s)} i^{(n)} \right) \times I \right) = I \left( v^{(s)} \right) \times I \left( i^{(n)} \right) \equiv T^{-s} \left( I \times I \left( \tilde{v}^{(s)} i^{(n)} \right) \right).
\]

Denote by \( \Sigma_n \) the \( \sigma \)-algebra generated by the fundamental intervals of order \( n \in \mathbb{N} \).

Let \( I_n(\Lambda) := \cup \cup_{\omega \in \Lambda} I \left( i^{(n)} \right), V_s(\Lambda) \equiv \cup \cup_{\omega \in \Lambda} I \left( v^{(s)} \right), \bar{t}_n(\Lambda) := \cup \cup_{\omega \in \Lambda} \bar{t}^{(n)}(\omega) \) for any \( \Lambda \subset X^n, \Lambda' \subset X^s \). Clearly, \( I_n(\Lambda) \) and \( V_s(\Lambda') \) are typical elements of \( \Sigma_n \) and \( \Sigma_s \), respectively. We define a set-function \( \overline{\mu} \) on \( \Sigma_n \times \Sigma_s \) by setting

\[
    \overline{\mu} \left( I_n(\Lambda) \times V_s(\Lambda') \right) = \mu \left( \bar{t}_n(\Lambda) \cap \{ v \in V_s(\Lambda') \} \right)
\]

so defined is uniquely determined. Clearly, \( \{ \Sigma_n \times \Sigma_s, n, s \in \mathbb{N} \} \) generates the Borel \( \sigma \)-algebra on \( I \times I \). One can extend the function \( \overline{\mu} \) to a measure on \( B_I \times B_I \), which we also denote \( \overline{\mu} \). By Caratheodory’s theorem such an extension exists and is unique. Let us denote by \( \overline{\lambda} \) the Lebesgue measure on \( I \times I \).
Theorem 1 (Properties of $\mu$).

i. $\mu$ is invariant under $T$ and $T^{-1}$;

ii. $\mu$ has marginal distributions equal to $\mu$;

iii. $\mu$ is a symmetric measure.

To prove the last assertion, we use the ergodic Theorem 5 in [2] or [9].

Theorem 1 implies that one can replace (3) by the symmetric relation in the definition of $\mu$.

By the Radon-Nikodym theorem there uniquely exists a measurable nonnegative random variable $\alpha$ on $\{\lambda \times \lambda\}$ such that

$$\mu(A) = \int_A \alpha d\lambda, \ A \in B_I \times B_I.$$

In the sequel the Hölder condition has the meaning defined in [4] while the kernel associated with a piecewise monotonic transformation has the meaning defined in [7].

Proposition 2 (Properties of $\alpha$).

i. $\int_0^1 \alpha(x, y) dy = h(x)$ for any $x \in I$, and $\alpha$ is symmetric;

ii. $\alpha$ satisfies a Hölder condition of order 1;

iii. $\alpha$ is a kernel.

We can now define the infinite order chains involving $T$ and $\mu$. Our definitions here are formally identical with those for the RCF expansion (see [4]).

We define the $X$-valued r.v.s. $\alpha_n, n \in \mathbb{Z}$, on $\{I \times I, B_I \times B_I\}$ by

$$\alpha_n(\theta, \omega) = a_n(\theta), n \in \mathbb{N}_+, \alpha_0(\theta, \omega) = a_0(\omega), \alpha_{-}(\theta, \omega) = a_{-}(\omega), \ l \in \mathbb{N}_+.$$

Hence $\alpha_n = \alpha_{n-1}(T), n \in \mathbb{Z}$.

We also consider the $I$-valued random variables $\gamma_l, l \in \mathbb{Z}$, defined by

$$\gamma_l(\theta, \omega) = \tau^l(\omega), l \in \mathbb{N}, \gamma_0(\theta, \omega) = s^0(\theta), n \in \mathbb{N}_+.$$

The doubly infinite sequences $(\alpha_n)_{n \in \mathbb{Z}}$ and $(\gamma_n)_{n \in \mathbb{Z}}$ are respectively $I \times I, X$- and $I$-valued strictly stationary symmetric processes under $\mu$. In other words, they are infinite order chains on $(I \times I, B_I \times B_I, \mu)$.

The introduction in the next section of probability measures $\mu_w, w \in I$, will allow us to complete the description of probabilistic properties of our infinite order chains. It is based on classical notions as given in [3].

3. CONDITIONAL PROBABILITY MEASURES

Whatever $w \in I$ define

$$\mu_w(A) = \int_A \alpha(x, w) dx, \ A \in B_I.$$

Then $\mu_w(\cdot)$ such defined is a probability on $B_I$. In the RCF case $\mu_w(\cdot)$ can be expressed in closed form and coincides with the function denoted by $\gamma^w(\cdot)$ in [4]; $\mu_w(\cdot)$ has most of its properties.
Theorem 3 (Properties of $\mu_w$).

i. $\mathbf{E}(\bar{a}_i = i | \bar{a}_0, \bar{a}_1, \ldots) = \mu_\alpha (I(i)) = P(\bar{a}_0, i) \quad \mathbf{P} - a.s., \ i \in X$;

ii. $\mathbf{E}(A | \bar{a}_0) = \mu_\alpha (A) \quad \mathbf{P} - a.s., \ A \in B_I$;

iii. $(\bar{a}_n)_{n \in \mathbb{Z}}$ is an $I$-valued Markov chain on $(I \times I, B_I \times B_I, \mathbf{P})$.

We now return to the random variables on $(I, B_I)$ involved in the conjecture. By the next theorem we see the impact of probabilistic properties of infinite order chains on the sequence $(s_n^w)_{n \in \mathbb{N}}$ defined on $(I, B_I, \mu_w)$. Below $E_w$ denotes the mean under $P_w$, $w \in I$.

Theorem 4 (Properties of the distribution of $s_n^w$).

i. $\lambda \equiv \mu_w, w \in I$;

ii. $\mu (A) = \int \mu_n (s_n^w \in A) \mu (dw), \ A \in B_I, \ n \in \mathbb{N}_*$;

iii. $\mu_w (s_n^w \in A) = E_w (\chi_A \{s_n^w\}) \equiv U^n \chi_A (w), \ w \in I, \ n \in \mathbb{N}_*$.

Hence, whatever $w \in I$, $(s_n^w)_{n \in \mathbb{N}}$ is an $I$-valued Markov chain on $(I, B_I, \mu_w)$ with transition operator $U$.

Using the results above we can prove the next result which is essentially used in the proof of Theorem 6 below.

Theorem 5. We have $\lambda = \mu_0$ and $\alpha (w, 0) = h (0) = \alpha (0, w), \ w \in I$.

4. THE SOLUTION

As we already said, our result confirming the conjecture stated in Section 1 concerns $f$-expansion satisfying Rényi’s condition on distortion and a strengthened Lipschitz condition. More precisely, we have

Theorem 6 (Main result). Consider an $f$-expansion for which conditions (E) and (C) hold.

i. Equations (2) are valid. Hence the label sequence $(a_n)_{n \in \mathbb{N}_0}$ under $\lambda = \mu_0$ is equivalent to the sequence $(\zeta_n)_{n \in \mathbb{N}}$ under $P_0$.

ii. The sequence $(s_n^0)_{n \in \mathbb{N}}$ on $(I, B_I, \lambda)$ is an $I$-valued homogenous Markov chain equivalent to the

Markov chain $(\zeta_n)_{n \in \mathbb{N}}$ on $(\Omega, K, P_0)$.

iii. The representation map $\varphi$ settles an isomorphism of measure spaces between $(I, B_I, \lambda)$ and $(\Omega, K, P_0)$.

The statement of conditions (E) and (C) can be found in [1],[8],[9].

Note that conditions (E) and (C) imply a condition slightly weaker than (BD(2)) (esssup replaces sup in (BD(2))) so that one can say that the conjecture is proved as formulated.
REFERENCES


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